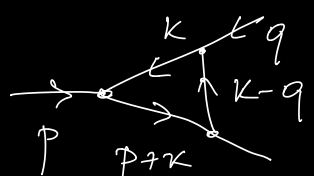
Two-loop wave function renormalization

- 1-loop diagrams



$$I_3 = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(k+p)^2} \frac{1}{(k-q)^2}$$

$$I_3 = \int \frac{d^dk}{(2\pi)^d} \, d_3x \, \frac{\Gamma(3)}{\left(k^2 x_1 + (p+k)^2 x_2 + (k-q)^2 x_3\right)^3}$$

$$= \int \frac{d^dk}{(2\pi)^d} \, d_3x \, \frac{\Gamma(3)}{\left(k^2 + \omega_3(p,q)\right)^3}$$

$$\omega_3(p,q) = p^2 x_2 + q^2 x_3 - (p x_2 - q x_3)^2$$

$$= p^2 x_1 x_2 + q^2 x_1 x_3 + (p+q)^2 x_1 x_2$$

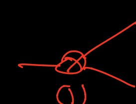
$$I_3 = \frac{\Gamma(3-\frac{d}{2})}{(4\pi)^{d/2}} \int d_3x \, \omega_3^{\frac{d}{2}-3}(p,q)$$

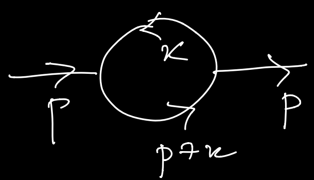
$$\underset{\varepsilon \to 0}{=} \frac{1}{(4\pi)^3} \frac{1}{2} \frac{1}{3-\frac{d}{2}} + O(1) = \frac{1}{(4\pi)^3} \frac{1}{\varepsilon}$$

$$\delta_g^{(1)} = - \frac{g^3}{(4\pi)^3} \frac{1}{\varepsilon}$$

$$g_0 = \mu^{\varepsilon/2} g$$

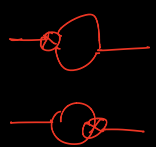
$$\delta_g^{(1)} = g \mu^{\varepsilon/2} \frac{g^2}{(4\pi)^3} \frac{1}{\varepsilon}$$

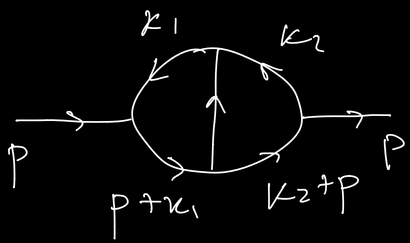




$$I_1 = \frac{(p^2)^{\frac{d}{2}-2}}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma(d-2)}$$

- Two loop diagram





$$(p^2)^{d-5} f(d)$$

$$\xrightarrow[\varepsilon\to 0]{} (p^2)(p^2)^{\varepsilon}\left(\frac{A_2}{\varepsilon^2}+\frac{A_1}{\varepsilon}\right)$$

$$p^2\left(\frac{A_2}{\varepsilon^2}+\frac{A_1}{\varepsilon}+\underline{\frac{A_2}{\varepsilon}\log p^2}\right)$$

$$I_2 = \int \frac{d^dk_2}{(2\pi)^d}\frac{d^dk_1}{(2\pi)^d} \frac{1}{k_1^2}\frac{1}{(k_1+p)^2}\frac{1}{(k_1-k_2)^2}\frac{1}{k_2^2}\frac{1}{(k_2+p)^2}$$

$$= \int d_3x \frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k+p)^2}\frac{\omega_3^{\frac{d}{2}-3}(p,k)}{(4\pi)^{d/2}}\Gamma\left(3-\frac{d}{2}\right)$$

$$= \frac{\Gamma\left(3-\frac{d}{2}\right)}{(4\pi)^{d/2}}\int d_3x \frac{d^dk}{(2\pi)^d} dy \frac{\omega_3^{\frac{d}{2}-3}(p,k)}{(k^2+2kpy+p^2y)^2}$$

$$= \frac{\Gamma\left(5-\frac{d}{2}\right)}{(4\pi)^{d/2}}\int d_3x\, dy\, dz \frac{d^dk}{(2\pi)^d}\frac{z^{2-\frac{d}{2}}(1-z)}{\left[z\,\omega_3(p,k)+(1-z)(k^2+2kpy+p^2y)\right]^{5-\frac{d}{2}}}$$

Denominator

$$D = k^2\left(z\, x_3(1-x_3) + 1 - z\right) + 2kp\left(z\, x_2 x_3 + (1-z)\, y\right)$$
$$+ p^2\left(z\, x_2(1-x_2) + (1-z)\, y\right)$$
$$= k^2 P_2 + 2kp\, P_1 + p^2 P_0$$

$$P_2\left(k^2 + 2kp\frac{P_1}{P_2}\right) + p^2 P_0$$
$$P_2\left(k + p\frac{P_1}{P_2}\right)^2 + p^2 P_0 - p^2\frac{P_1^2}{P_2}$$

As a result,

$$I_2 = \frac{\Gamma(5-d)}{(4\pi)^d}\left(p^2\right)^{d-5} \int d_3x\, dy\, dz \;\; z^{2-\frac{d}{2}}(1-z)$$

$$z^{2-3} = \frac{1}{z}$$

$$\times \frac{\left(P_0 - \frac{P_1^2}{P_2}\right)^{d-5}}{P_2^{d/2}}$$

As usual there are singularities

$z=0$ is not integrable for $d=6$.

Unfortunatly this is not all:

$P_2 = 0$ is potentially dangerous

$$P_2 = z\, x_3(1-x_3) + 1 - z$$

$z = 1$ and $x_3 = 0$ or $x_3 = 1$

$$P_2(z,x_3) = 1 - z + z\, x_3 (1 - x_3)$$

$$P_1(z,x_3) = (1-z) y + z\, x_2 x_3$$

$$P_0(z,x_3) = (1-z) y + z\, x_2 (1 - x_2)$$

Evaluate at dangerous points $z = 1$:

$$P_2(1, x_3) = x_3 (1 - x_3)$$

$$\int d_3 x = \int_0^1 dx_3 \int_0^{1-x_3} dx_2$$

$$P_1(1, x_3) = x_2 x_3$$

$$P_0(1, x_3) = x_2 (1 - x_2)$$

Therefore, $\dfrac{P_1^2(1, x_3)}{P_2(1, x_3)} = \dfrac{x_2^2 x_3^2}{x_3(1-x_3)}$

is regular both for $x_3 \to 0$ and $x_3 \to 1$

$$\tilde{I}_2 = \int d_3 x\, dy\, dz\; z^{2-\frac{d}{2}} \underbrace{(1-z) \frac{\left(P_0 - \frac{P_1^2}{P_2}\right)^{d-5}}{P_2^{d/2}}}_{f_1(z,x_3)}$$

$$= \int d_3 x\, dy\, dz\; z^{2-\frac{d}{2}} f_1(z, x_3)$$

$$= \int d_3 x\, dy\, dz\; z^{2-\frac{d}{2}} \left[f_1(0, x_3) + f_1(z, x_3) - f_1(0, x_3) \right]$$

$$z^{2-\frac{d}{2}} \left[f_1(z, x_3) - f_1(0, x_3) \right] \to \frac{f_2(z, x_3)}{P_2^{d/2}(z, x_3)}$$

$$= \int d_3x\, dy\, dz\, z^{2-\frac{d}{2}} f_1(0, x_3) + \int d_3x\, dy\, dz\, \frac{f_2(z, x_3)}{P_2^{d/2}(z, x_3)}$$

$$= C_{P0} + \int d_3x\, dy\, dz\, \frac{f_2(1, x_3) + \frac{\partial f_2}{\partial z}(1, x_3)(z-1)}{P_2^{d/2}(z, x_3)}$$

$$+ \int d_3x\, dy\, dz\, \frac{f_2(z, x_3) - f_2(1, x_3) - \frac{\partial f_2}{\partial z}(1, x_3)(z-1)}{P_2^{d/2}} \Bigg|_{d=6}$$

$$= C_{P0} + C_{F1} + \underbrace{\int dx_3\, f_3(x_3)}_{\text{not integrable for } d=6}$$

Asymptotic behavior

$$f_3(x_3) \underset{x_3 \to 0}{=} \frac{A}{x_3^{d/2-2}}, \qquad A = \frac{4\,\Gamma^2(d-4)}{(d-2)(d-4)\,\Gamma(2d-8)}$$

Then

$$\tilde{I}_2 = C_{P0} + C_{F1} + \underbrace{\int dx_3\, \frac{A}{x_3^{d/2-2}}}_{C_{P1}} + \underbrace{\int dx_3 \left[f(x_3) - \frac{A}{x_3} \right]}_{C_{F2}} \Bigg|_{d=6}$$

$$= C_{P0} + C_{F1} + C_{P1} + C_{F2}$$

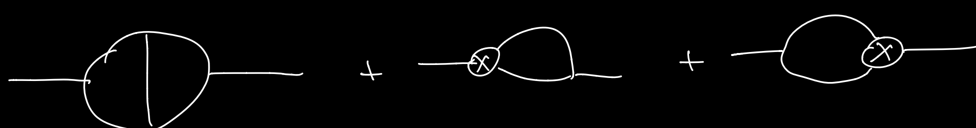
$$\underset{\varepsilon \to 0}{=} \frac{1}{3\varepsilon} + \frac{2}{\varepsilon} + O(\varepsilon)$$

$$I_2 = \frac{1}{(4\pi)^6}\left[-\frac{p^2}{3\varepsilon^2} + \frac{p^2}{3\varepsilon}\left(\gamma - \ln 4\pi + \ln p^2 - 3\right) + O(1)\right]$$

We will also use

$$I_1 = \frac{1}{(4\pi)^3}\left[-\frac{p^2}{3\varepsilon} + \frac{p^2}{6}\left(\gamma - \ln 4\pi + \ln p^2 - \frac{8}{3}\right) + O(1)\right]$$

Adding all terms we get



$$\frac{1}{2}(-g)^4 I_2 + \frac{1}{2}(-g)\left(-\delta_g^{(1)}\right) I_1 \cdot 2$$

$$= \frac{g^4}{(4\pi)^6}\Bigg[-\frac{p^2}{6\varepsilon^2} + \frac{p^2}{6\varepsilon}(\beta - 3)$$

$$+ \frac{p^2}{3\varepsilon^2} - \frac{p^2}{6\varepsilon}\left(\beta - \frac{8}{3}\right)\Bigg]$$

$$= \frac{g^4}{(4\pi)^6}\left[\frac{p^2}{6\varepsilon^2} - \frac{p^2}{18\varepsilon}\right]$$

$f(x)$ regular : $f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$

$d = 4 - \varepsilon$:

$$\int \frac{f(x)}{x^{d/2}} dx \,, \qquad \frac{f(x)}{x^{d/2}} \underset{x \to 0}{=} \frac{f_0 + f_1 x + f_2 x^2}{x^{d/2}} + \cdots =$$

$$= \frac{f_0}{x^{d/2}} + \frac{f_1}{x^{d/2-1}} + \frac{f_2}{x^{d/2-2}} + \cdots$$

$$\int \frac{1}{x^{d/2}} dx = \int_0^1 x^{-d/2} dx = \frac{1}{1-d/2} x^{1-\frac{d}{2}} \Big|_0^1 = \frac{1}{1-d/2}$$

$d < 2$

$$\int \frac{f(x) - f_0 - f_1 x - f_2 x^2}{x^{d/2}} dx + \int \frac{\left(f_0 + f_1 x + f_2 x^2\right)}{x^{d/2}} dx$$

$$\frac{f_3 x^3}{x^{d/2}} \to \frac{f_3 x^3}{x^2} = f_3 x$$

$$\int \frac{f(x) - f_0 - f_1 x - f_2 x^2}{x^2} dx + \text{ easy for arbitrary } d$$

$$\int \frac{f_0}{x^{d/2}} dx = \frac{f_0}{1-d/2}$$

$$\int \frac{f_1}{x^{d/2-1}} dx = \frac{f_0}{2-d/2}$$

$$I = \int d^d k_1\, d^d k_2 \; \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{(k_1+p)^2} \frac{1}{(k_2+p)^2} \frac{1}{(k_1-k_2)^2}$$

$$\int \frac{1}{k_1^{2\alpha_1}} \frac{1}{k_2^{2\alpha_2}} \frac{1}{(k_1+p)^{2\beta_1}} \frac{1}{(k_2+p)^{2\beta_2}} \frac{1}{(k_1-k_2)^{2\gamma}} = I(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma)$$

$$I d = \int \underbrace{\frac{\partial (k_{1\mu} - k_{2\mu})}{\partial k_{1\mu}}}_{\delta_{\mu\mu} = d} \frac{1}{k_1^2} \frac{1}{k_2^2} \cdots \frac{1}{(k_1-k_2)^2}$$

$$= - \int (k_1 - k_2)_\mu \frac{\partial}{\partial k_{1\mu}} \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{(k_1+p)^2} \frac{1}{(k_2+p)^2} \frac{1}{(k_1-k_2)^2}$$

$$= \int \frac{1}{k_1^2} \frac{1}{k_2^2} \frac{1}{(k_1+p)^2} \frac{1}{(k_2+p)^2} \frac{1}{(k_1-k_2)^2}$$

$$\left[\frac{2 k_{1\mu} (k_1 - k_2)_\mu}{k_1^2} + \frac{2 (k_1+p)(k_1-k_2)}{(k_1+p)^2} + \frac{2(k_1-k_2)^2}{(k_1-k_2)^2} \right]$$

$$= \int (\quad) \left[2 + 2 + \frac{(k_1-k_2)^2 - k_1^2 - k_2^2}{k_1^2} \right.$$

$$\left. + \frac{2(k_1+p)(k_1+p-k_2-p)}{(k_1+p)^2} \right]$$

:nit propagators

$$= \int (\quad) \left[4 + \frac{(k_1-k_2)^2 - k_2^2}{k_1^2} + \frac{(k_1-k_2)^2 - (k_2+p)^2}{(k_1+p)^2} \right]$$

$$= 4I + I(2,1,1,1,0) - I(2,0,1,1,1)$$

$$+ I(1,1,2,1,0) - I(1,1,2,0,1)$$

$k_1 \leftrightarrow k_2$ or $\begin{pmatrix} k_1 \leftrightarrow p+k_1 \\ k_2 \leftrightarrow p+k_2 \end{pmatrix}$

$$I(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma) = I(\alpha_2, \alpha_1, \beta_2, \beta_1, \gamma)$$

$$= I(\beta_1, \beta_2, \alpha_1, \alpha_2, \gamma)$$

$$I \equiv I(1,1,1,1,1) = \frac{2}{4-d}\left[I(2,0,1,1,1) - I(2,1,1,1,0)\right]$$

$$I(2,1,1,1,0) = \int \frac{1}{k_1^4} \frac{1}{(k_1+p)^2} \frac{1}{k_2^2} \frac{1}{(k_2+p)^2} = \frac{(p^2)^{d-5}}{(4\pi)^d} f(1,1) f(2,1)$$

$$I(2,0,1,1,1) = \int \frac{1}{k_1^4} \frac{1}{(k_1+p)^2} \frac{1}{(k_2+p)^2} \frac{1}{(k_1-k_2)^2} = \frac{(p^2)^{d-5}}{(4\pi)^d} f(1,1) \cdot f\left(2, 3-\frac{d}{2}\right)$$

$$\Downarrow$$

$$(k_1+p)^{\#}$$

$$\int \frac{d^d k}{(2\pi)^{d/2}} \frac{1}{k^{2\lambda_1}(k+p)^{2\lambda_2}} = \frac{(p^2)^{\frac{d}{2}-\lambda_1-\lambda_2}}{(4\pi)^{d/2}} f(\lambda_1, \lambda_2)$$